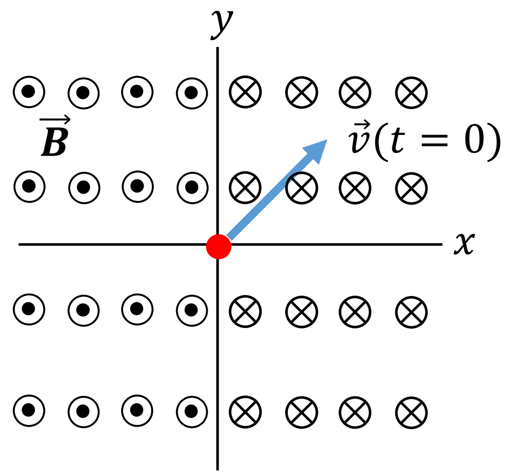
PHYS 102 Midterm Exam 2 Solution 2020-21-2

1. As shown in the figure, the magnetic field changes sign in a region of space, so it can be expressed as:



$$\vec{\boldsymbol{B}}(x,y) = \begin{cases} B_0\,\hat{\mathbf{k}}, & x < 0 \\ -B_0\,\hat{\mathbf{k}}, & x \geq 0 \end{cases}.$$

A particle of mass m and charge $Q > 0$ is at the origin at $t = 0$, and has velocity

$$\vec{\boldsymbol{v}}(t=0) = \frac{v_0}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}}).$$

(a) (7 Pts.) How long will it take for this particle to cross the y-axis (i.e. $x = 0$)?

(b) (7 Pts.) How long will it take for the velocity of the particle to come back to its initial value?

(c) (7 Pts.) What will be the average velocity of the particle when its velocity is back to its initial velocity?

(d) (4 Pts.) In which direction (give a unit vector) will the particle move on average if its initial velocity was

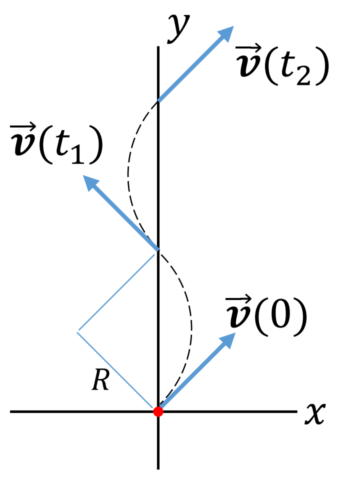
$$\vec{\boldsymbol{v}}(t=0) = \frac{v_0}{\sqrt{2}}(\hat{\mathbf{i}} - \hat{\mathbf{j}})\ ?$$

Solution: (a) The force on the particle at the instant it enters the magnetic field is

$$\vec{\boldsymbol{F}} = Q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}} = \frac{Qv_0B_0}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}}) \times (-\hat{\mathbf{k}}) = \frac{Qv_0B_0}{\sqrt{2}}(-\hat{\mathbf{i}} + \hat{\mathbf{j}})\,.$$

Since $|\vec{\boldsymbol{v}}| = v_0$ is constant during the motion, the path of the particle is a circle with radius $R = (mv_0)/(QB_0)$. Because the particle enters the field at an angle $\pi/4$, it will move along a quarter circle and cross the y-axis in time $T/4$, where T is the period for the full circle. Therefore, time t_1 it takes for this particle to cross the y-axis is

$$t_1 = \frac{1}{4}\frac{2\pi R}{v_0} = \frac{\pi m}{2QB_0}.$$



(b) The particle will enter the region $x < 0$ with velocity $\vec{\boldsymbol{v}}(t_1) = v_0(-\hat{\mathbf{i}} + \hat{\mathbf{j}})/\sqrt{2}$. Therefore, at that instant time magnetic force on the particle will be

$$\vec{\boldsymbol{F}} = \frac{Qv_0B_0}{\sqrt{2}}(-\hat{\mathbf{i}} + \hat{\mathbf{j}}) \times \hat{\mathbf{k}} = \frac{Qv_0B_0}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}})\,.$$

In this region the path of the particle will again be a quarter circle with the same radius. Therefore, the particle will cross the y-axis once more, this time into the region $x > 0$, and its velocity will come back to its initial value. Hence, the time it takes will be

$$t_2 = 2t_1 = \frac{\pi m}{QB_0}.$$

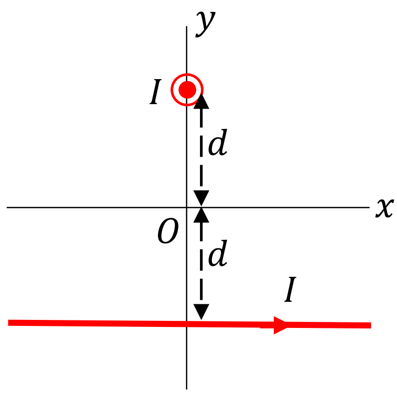
(c) The displacement of the particle during the time interval t_2 will be

$$\Delta\vec{\boldsymbol{r}} = 2\sqrt{2}R\,\hat{\mathbf{j}} = 2\sqrt{2}\frac{mv_0}{QB_0}\,\hat{\mathbf{j}}\,,$$

therefore, its average velocity will be $\vec{\boldsymbol{v}}_{\text{av}} = \Delta\vec{\boldsymbol{r}}/t_2 = 2\sqrt{2}v_0/\pi$.

(d) In this case the force on the particle at time $t = 0$ will be $\vec{\boldsymbol{F}} = \frac{Qv_0B_0}{\sqrt{2}}(\hat{\mathbf{i}} - \hat{\mathbf{j}}) \times (-\hat{\mathbf{k}}) = \frac{Qv_0B_0}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$. Its average velocity will again be in the $+\hat{\mathbf{j}}$-direction, but in each region the path will be three quarters of the circle.

2. An infinitely long straight wire placed at $y = -d$ carries a current I in the $+x$-direction, and another infinitely long straight wire placed at $y = +d$ carries the same current I in the $+z$-direction, as shown in the figure.



(a) (20 Pts.) Using the coordinate system indicated in the figure, find the magnetic field on the x-axis as a function of x.

(b) (5 Pts.) At which point does the magnitude of the magnetic field on the x-axis attain its maximum value?

Solution:

(a) For an infinitely long straight wire, we have

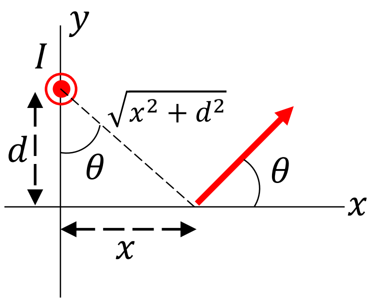
$$\oint \vec{B}\cdot d\vec{\ell} = \oint B(r)d\ell = B(r)\oint d\ell = B(r)(2\pi r) = \mu_0 I \quad \rightarrow \quad B(r) = \frac{\mu_0 I}{2\pi r},$$

where r is the perpendicular distance from the wire to the point at which the magnetic feld magnitude is evaluated.

Every point on the x-axis is at the same perpendicular distance d from the current at $y = -d$. Therefore, we have

$$\vec{B}_1 = \frac{\mu_0 I}{2\pi d}\,\hat{\mathbf{k}}\,.$$

The magnetic field produced by the current at $y = +d$ is found from the following figure as



$$\vec{B}_2 = B_{2x}\,\hat{\mathbf{i}} + B_{2y}\,\hat{\mathbf{j}} = \frac{\mu_0 I}{2\pi\sqrt{x^2+d^2}}(\cos\theta\ \hat{\mathbf{i}} + \sin\theta\ \hat{\mathbf{j}})\,.$$

Since

$$\cos\theta = \frac{d}{\sqrt{x^2+d^2}}, \qquad \sin\theta = \frac{x}{\sqrt{x^2+d^2}},$$

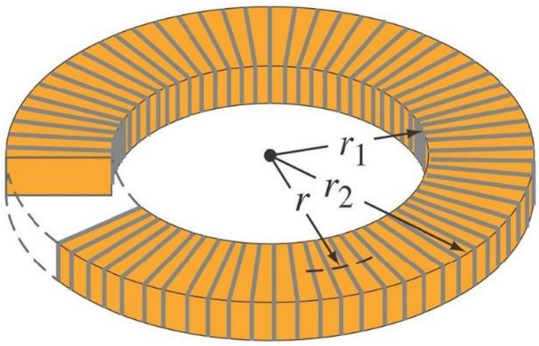
$$\vec{B} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0 I}{2\pi(x^2+d^2)}(d\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}) + \frac{\mu_0 I}{2\pi d}\,\hat{\mathbf{k}}\,.$$

(b) The magnitude is

$$B = \frac{\mu_0 I}{2\pi}\sqrt{\frac{1}{x^2+d^2} + \frac{1}{d^2}} = \frac{\mu_0 I}{2\pi d}\sqrt{\frac{x^2+2d^2}{x^2+d^2}},$$

which clearly is maximum at $x = 0$.

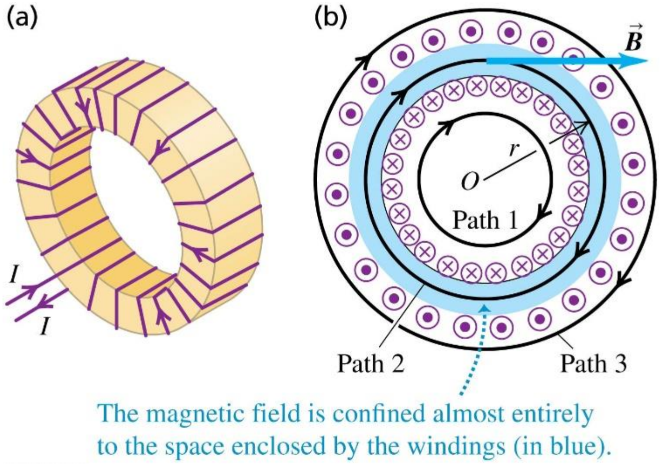
3. (25 Pts.) The figure shows a doughnut-shaped toroidal solenoid with inner radius r_1 and outer radius r_2, tightly wound with N turns of wire carrying a current I. Find the magnitude of the magnetic field for $r < r_1, r_1 < r < r_2$, and $r > r_2$.



Solution: (Example 28.10 solved in the textbook)

The symmetry of the problem tells us that the magnetic field lines must be circles concentric with the toroid axis. Therefore we choose circular integration paths for use with Ampère's law, so that the magnetic field (if any) is tangent to each path and its magnitude is constant at all points along the path.

Therefore, along each path



$$\oint \vec{B} \cdot d\vec{\ell} = B(r)(2\pi r) = \mu_0 I_{\text{enc}}.$$

The otal current enclosed by path 1 ($0 < r < r_1$) is zero, so we find

$$\vec{\boldsymbol{B}}(r) = 0, \qquad 0 < r < r_1 .$$

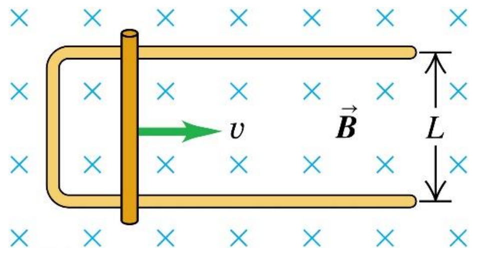
For path 2 ($r_1 < r < r_2$), each turn of the winding passes once through the area bolunded by this path, so $I_{\text{enc}} = NI$. Therefore,

$$B(r) = \frac{\mu_0 NI}{2\pi r}, \qquad r_1 < r < r_2 .$$

For path 3 ($r_2 < r < \infty$), each turn of the winding passes twice carrying the current in opposite directions through the area bolunded by this path, so $I_{\text{enc}} = 0$. Therefore,

$$\vec{\boldsymbol{B}}(r) = 0, \qquad r_2 < r < \infty .$$

4. A rectangular loop with width L and a slide wire with mass m are as shown in figure. A uniform magnetic field $\vec{\boldsymbol{B}}$ is directed perpendicular to the plane of the loop into the plane of the figure. The slide wire is given an initial speed of v_0 and then released. There is no friction between the slide wire and the loop, and the resistance of the loop is negligible in comparison to the resistance R of the slide wire.



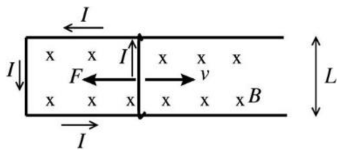
(a) (7 Pts.) Obtain an expression for F, the magnitude of the force exerted on the wire while it is moving at speed v.

(b) (10 Pts.) Find the distance d that the wire moves before coming to rest.

(c) (8 Pts.) Find the total energy dissipated by the resistance during the motion.

Solution:

(a) When the wire has speed v the induced emf is $\mathcal{E} = BLv$ and the induced current is $I_{\text{ind}} = \mathcal{E}/R = BLv/R$. According to Lenz's law, the induced current should be upward in the wire so that the force $\vec{\boldsymbol{F}} = I_{\text{ind}}\vec{\boldsymbol{L}} \times \vec{\boldsymbol{B}}$ is to the left, opposing the motion. Magnitude of the force is



$$F = I_{\text{ind}}LB \quad \rightarrow \quad F = \frac{B^2L^2v}{R}.$$

(b) Taking the positive direction to the right, and using Newton's second law, we have

$$a = \frac{dv}{dt} = -\frac{F}{m} \quad \rightarrow \quad \frac{dv}{dt} = -\frac{B^2L^2v}{m\,R}.$$

Integrating the last equation, speed of wire is found as

$$\frac{dv}{dt} = -\frac{B^2L^2v}{m\,R} \quad \rightarrow \quad \int_{v_0}^{v}\frac{dv'}{v'} = -\frac{B^2L^2}{m\,R}\int_0^t dt' \quad \rightarrow \quad v = v_0e^{-\left(\frac{B^2L^2}{m\,R}\right)t}.$$

Wire comes to rest when $t \rightarrow \infty$. Therefore, the distance d that the wire moves before coming to rest is found as

$$v = \frac{dx}{dt} = v_0e^{-\left(\frac{B^2L^2}{m\,R}\right)t} \quad \rightarrow \quad d = v_0\int_0^{\infty}e^{-\left(\frac{B^2L^2}{m\,R}\right)t}\,dt \quad \rightarrow \quad d = \frac{mRv_0}{B^2L^2}.$$

(c) The wire stops when all its initial kinetic energy is dissipated by the resistance in the circuit. Hence $\Delta E = mv_0^2/2$.

This can also be found using the power dissipated in the resistor as follows:

$$P = RI_{\text{ind}}^2 = \frac{B^2L^2v^2}{R} = \frac{B^2L^2v_0^2}{R}e^{-2\left(\frac{B^2L^2}{m\,R}\right)t} \quad \rightarrow \quad \Delta E = \int_0^{\infty}P\,dt = \frac{B^2L^2v_0^2}{R}\int_0^{\infty}e^{-2\left(\frac{B^2L^2}{m\,R}\right)t}\,dt = \frac{1}{2}mv_0^2.$$